ANOTHER APPROACH TO PARAMETRIC BING AND KRASINKIEWICZ MAPS

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ABSTRACT. Using a factorization theorem due to Pasynkov [11] we provide a short proof of the existence and density of parametric Bing and Krasinkiewicz maps. In particular, the following corollary is established: Let $f\colon X \to Y$ be a surjective map between paracompact spaces such that all fibers $f^{-1}(y)$, $y \in Y$, are compact and there exists a map $g\colon X \to \mathbb{I}^{\aleph_0}$ embedding each $f^{-1}(y)$ into $\mathbb{I}^{\aleph_0}$. Then for every $n \geq 1$ the space $C^*(X, \mathbb{R}^n)$ of all bounded continuous functions with the uniform convergence topology contains a dense set of maps g such that any restriction $g|f^{-1}(y)$, $y \in Y$, is a Bing and Krasinkiewicz map.

1. Introduction

All spaces in the paper are assumed to be paracompact and all maps continuous. All maps from X to M are denoted by $C(X, M)$. Usually, $C(X, M)$ will carry either the uniform convergence topology or the source limitation topology. When X is compact, these two topologies coincide. Unless stated otherwise, a space (resp., compactum) means a metrizable space (resp., compactum).

In this paper we provide another approach to prove results concerning parametric Bing and Krasinkiewicz maps. The approach is based on Pasynkov's technique developed in [10] and [11].

Bing maps and Krasinkiewicz maps have been extensively studied recent years (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [13], [14]). Recall that a map f between compact spaces is said to be a *Bing map* [4] provided all fibers of f are Bing spaces. Here, a compactum is a *Bing space* if each of its subcontinua is indecomposable. Following Krasinkiewicz [2], we say that a space M is a *free space* if for any compactum X the function space $C(X, M)$ contains a dense subset consisting of Bing maps. The class of free spaces is quite large, it contains all n-dimensional

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manifolds ($n \geq 1$) [2], the unit interval [4], all locally finite polyhedra [12], all manifolds modeled on the Menger cube M^n_{2n+1} or the Nöbeling space N^n_{2n+1} [12], as well as all 1-dimensional locally connected continua [12].

Next theorem follows from the proof of [13, Theorem 1.2] where the special case with X and Y being metrizable was established.

Theorem 1.1. *Let M be a free ANR-space and $f\colon X \to Y$ be a perfect map with $W(f) \leq \aleph_0$, where X and Y are paracompact. Then the maps $g \in C(X, M)$ such that all restrictions $g|f^{-1}(y)$, $y \in Y$, are Bing maps form a dense set $B \subset C(X, M)$ with respect to the source limitation topology. Moreover, B is G_δ provided Y is first countable.*

Here, $W(f) \leq \aleph_0$ (see [10]) means that there exists a map $g\colon X \to \mathbb{I}^{\aleph_0}$ such that $f\triangle g\colon X \to Y \times \mathbb{I}^{\aleph_0}$ is an embedding. For example [10, Proposition 9.1], $W(f) \leq \aleph_0$ for any closed map $f\colon X \to Y$ such that X is a metrizable space and every fiber $f^{-1}(y)$, $y \in Y$, is separable.

Although, the arguments from [13] don't work when the map f in Theorem 1.1 is not perfect or the space M is not ANR, we have the following result:

Theorem 1.2. *Let X and Y be paracompact spaces and $f\colon X \to Y$ be a map with compact fibers and $W(f) \leq \aleph_0$. Then for every compact free space M the space $C(X, M)$ equipped with the uniform convergence topology contains a dense subset of maps g such that all restrictions $g|f^{-1}(y)$, $y \in Y$, are Bing maps.*

The second type of results concern Krasinkiewicz maps. A space M is said to be a *Krasinkiewicz space* [9] if for any compactum X the function space $C(X, M)$ contains a dense subset of Krasinkiewicz maps. Here, a map $g\colon X \to M$, where X is compact, is said to be Krasinkiewicz [5] if every continuum in X is either contained in a fiber of g or contains a component of a fiber of g. The class of Krasinkiewicz spaces contains all Euclidean manifolds and manifolds modeled on Menger or Nöbeling spaces, all polyhedra (not necessarily compact), as well as all cones with compact bases (see [3], [5], [6], [8], [9]).

Theorem 1.3. *Let $f\colon X \to Y$ be a map with compact fibers and $W(f) \leq \aleph_0$, where X and Y are paracompact spaces. If M is a compact Krasinkiewicz space, then $C(X, M)$ equipped with the uniform convergence topology contains a dense subset of maps g such that all restrictions $g|f^{-1}(y)$, $y \in Y$, are Krasinkiewicz maps.*

Corollary 1.4. *Let $f\colon X \to Y$ be a map with compact fibers such that $W(f) \leq \aleph_0$, where X and Y are paracompact spaces. Then for every*

$n \geq 1$ the space $C^*(X, \mathbb{R}^n)$ of all bounded continuous functions with the uniform convergence topology contains a dense set of maps g such that any $g|f^{-1}(y)$, $y \in Y$, is a Bing and Krasinkiewicz map.

Theorem 1.3 was established in [13, Theorem 1.1] for an arbitrary Krasinkiewicz ANR-space M in the case X, Y are metrizable, f is perfect and $C(X, M)$ is equipped with the source limitation topology. Let us note that the proof of [13, Theorem 1.1] provides the following result: Let $f\colon X \to Y$ be a perfect map between paracompact spaces with $W(f) \leq \aleph_0$, and let M be a Krasinkiewicz ANR-space. Then the maps $g \in C(X, M)$ such that all $g|f^{-1}(y)$, $y \in Y$, are Krasinkiewicz maps form a dense subset of $C(X, M)$ with respect to the source limitation topology. Moreover, this set is G_δ if Y is first countable.

Remark. The requirement in Theorems 1.2 - 1.3 f to have compact fibers is necessary because of the definition of Bing and Krasinkiewicz maps. If we define a Bing space to be a space such that any its subcontinuum (a connected compactum) is indecomposable, and a Bing map to be a map whose fibers are Bing spaces, then Theorem 1.2 remains valid for any map f with $W(f) \leq \aleph_0$. Then same remark is true for Theorem 1.3 if by a Krasinkiewicz map we mean any map $g\colon X \to M$, where X is not necessary compact, such that every continuum in X is either contained in a fiber of g or contains a component of a fiber of g.

2. Bing and Krasinkiewicz maps

This section contains the proof of Theorem 1.2, Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.2. Since $W(f) \leq \aleph_0$, there exists a map $\lambda\colon X \to \mathbb{I}^{\aleph_0}$ such that $f \triangle \lambda\colon X \to Y \times \mathbb{I}^{\aleph_0}$ is an embedding. We also fix a map $g_0\colon X \to M$ and a number $\epsilon > 0$. We are going to find a map $g \in C(X, M)$ such that g is ϵ-close to g_0 and all restrictions $g|f^{-1}(y)$, $y \in Y$, are Bing maps. To this end, let $\overline{\lambda}\colon \beta X \to \mathbb{I}^{\aleph_0}$ and $\overline{g}_0\colon \beta X \to M$ be the Stone-Cech extensions of the maps λ and g_0, respectively. Then $\overline{\lambda} \triangle \overline{g}_0 \in C(\beta X, \mathbb{I}^{\aleph_0} \times M)$. We consider also the constant maps $\eta_1\colon \mathbb{I}^{\aleph_0} \times M \to Pt$ and $\eta_2\colon \beta Y \to Pt$, where Pt is the one-point space. According to Pasynkov's factorization theorem [11, Theorem 13], there exist metrizable compacta K, T and maps $f^*\colon K \to T$, $\xi_1\colon \beta X \to K$, $\xi_2\colon K \to \mathbb{I}^{\aleph_0} \times M$ and $\eta^*\colon \beta Y \to T$ such that:

- $\eta^* \circ \beta f = f^* \circ \xi_1$;
- $\xi_2 \circ \xi_1 = \overline{\lambda} \triangle \overline{g}_0$;

If $p\colon \mathbb{I}^{\aleph_0} \times M \to \mathbb{I}^{\aleph_0}$ and $q\colon \mathbb{I}^{\aleph_0} \times M \to M$ denote the corresponding projections, we have

$$p \circ \xi_2 \circ \xi_1 = \overline{\lambda} \text{ and } q \circ \xi_2 \circ \xi_1 = \overline{g}_0. \tag{1}$$

Since M is a free space, there exists a Bing map $\phi\colon K \to M$ such that ϕ is ϵ-close to $q \circ \xi_2$. Then the map $\overline{g} = \phi \circ \xi_1$ is ϵ-close to $\overline{g}_0$. Hence, the maps $g = \overline{g}|X$ and g_0 are also ϵ-close. According to (1), we have $\lambda = \big(p \circ \xi_2 \circ \xi_1\big)|X$. This implies that ξ_1 embeds each fiber $f^{-1}(y)$, $y \in Y$, into K (recall that λ embeds the fibers $f^{-1}(y)$ into $\mathbb{I}^{\aleph_0}$). Consequently, $f^{-1}(y) \cap g^{-1}(z)$ is homeomorphic to a subset of $\phi^{-1}(z)$ for all $z \in M$. Since the fibers $\phi^{-1}(z)$, $z \in M$, are Bing spaces, so are the spaces $f^{-1}(y) \cap g^{-1}(z)$. Therefore, each restriction $g|f^{-1}(y)$, $y \in Y$, is a Bing map. □

Proof of Theorem 1.3. We follow the notations and the arguments from the proof of Theorem 1.2. The only difference now is that M is a Krasinkiewicz space. So, there exists a Krasinkiewicz map $\phi\colon K \to M$ such that ϕ is ϵ-close to $q \circ \xi_2$. Then map $g \in C(X, M)$ is ϵ-close to g_0 and $f^{-1}(y) \cap g^{-1}(z)$ is homeomorphic to a compact subset of $\phi^{-1}(z)$ for all $y \in Y$ and $z \in M$. Since ϕ is a Krasinkiewicz map on K, for every continuum $C \subset f^{-1}(y)$ we have either C is contained in $\phi^{-1}(z)$ or contains a component of $\phi^{-1}(z)$ for some $z \in M$. This implies that C is either contained in $f^{-1}(y) \cap g^{-1}(z)$ or contains a component of $f^{-1}(y) \cap g^{-1}(z)$ for some $z \in M$. Therefore, any restriction $g|f^{-1}(y)$ is a Krasinkiewicz map. □

Proof of Corollary 1.4. Let $g_0\colon X \to \mathbb{R}^n$ and let $\overline{g}_0\colon \beta X \to \mathbb{R}^n$ be its Stone-Cech extension. Proceeding as above, we need the following fact (see [4] and [5], or [13]): if M is a free Krasinkiewicz space, then the maps $g \in C(K, M)$ which are both Bing and Krasinkiewicz form a dense G_δ-subset of $C(K, M)$. Since $\mathbb{R}^n$ is both a free space and a Krasinkiewicz space, we can choose a Bing and Krasinkiewicz map $\phi \in C(K, \mathbb{R}^n)$ which is ϵ-close to $\overline{g}_0$. Then any restriction map $g|f^{-1}(y)$ is also Bing and Krasinkiewicz. □

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